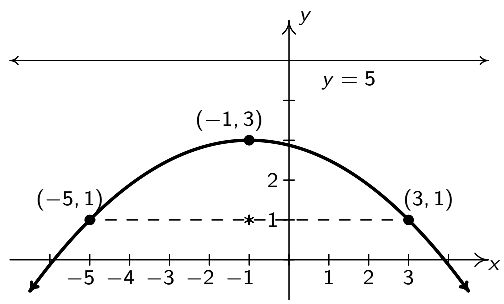
PARABOLAS

EXAMPLE:

1. (a) Rewriting $(x+1)^2 = -8(y-3)$ as $(x-(-1))^2 = -8(y-3)$, we identify $h=-1$ and $k=3$ so the vertex is $(-1,3)$. Since the x is squared but the y is not, we know we have a vertical parabola. Since $4p=-8$ so $p=-2$. Since $p<0$, the focus is *below* the vertex so the parabola opens *downwards*.

   The focal length is $|p|=2$, which means the focus is 2 units below the vertex. From $(-1,3)$, we move down 2 units and find the focus at $(-1,1)$. Likewise the directrix is 2 units above the vertex, or the horizontal line $y=5$.

   The focal diameter is $|4p|=|-8|=8$, which means the parabola is 8 units wide at the focus. Hence, the endpoints of the latus rectum are 4 units to the left and right of the focus. Starting at $(-1,1)$ and moving to the left 4 units, we arrive at $(-5,1)$. Starting at $(-1,1)$ and moving to the right 4 units we arrive at $(3,1)$. The final graph appears below.



The graph of $(x+1)^2 = -8(y-3)$

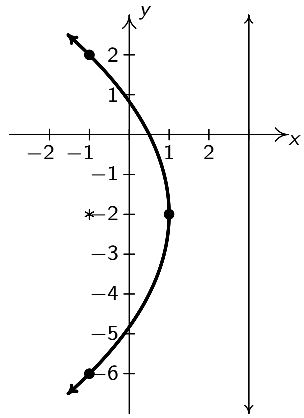
   (b) We start by rewriting $y^2+4y+8x=4$ in standard form.

$$\begin{aligned}
y^2+4y+8x &= 4 \\
y^2+4y &= -8x+4 \\
y^2+4y+4 &= -8x+4+4 && \text{Complete the Square in } y. \\
(y+2)^2 &= -8x+8 && \text{Factor the Perfect Square.} \\
(y+2)^2 &= -8(x-1) && \text{Factor out the coefficient of } x. \\
(y-(-2))^2 &= -8(x-1) && \text{Rewrite.}
\end{aligned}$$

   Identifying $h=1$ and $k=-2$, we get the vertex is $(1,-2)$. Since $4p=-8$, we get $p=-2$. The fact that $p<0$, means the focus will be the *left* of the vertex so the parabola will open to the *left*.

   Since the focal length is $|p|=2$, the focus is 2 units to the left of the vertex. From $(1,-2)$ and move left 2 units and arrive at the focus $(-1,-2)$. Similarly, the directrix is 2 units to the right of the vertex, the vertical line $x=3$.

   Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus. Starting at the focus $(-1,-2)$ we move down 4 units and get $(-1,-6)$. Moving up 4 units from the focus we get $(-1,2)$. Hence, $(-1,-6)$ and $(-1,2)$ are the endpoints of the latus rectum. The final graph appears below.



The graph of $y^2 + 4y + 8x = 4$

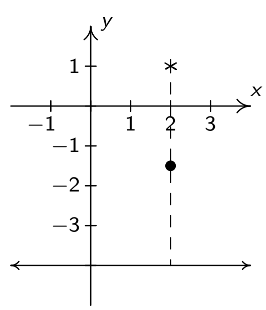
2. To represent the parabola $y^2 + 4y + 8x = 4$ as the graphs of two or more functions of x, we need to solve for y in terms of x. The equation $y^2 + 4y + 8x = 4$ is a quadratic equation in the variable y, which means we can use the quadratic formula to solve it. We arrange the equation to get it equal to zero: $y^2 + 4y + 8x - 4 = 0$. Since we are solving for y, we identify $a = 1$, $b = 4$ and $c = 8x - 4$. We find the discriminant $b^2 - 4ac = (4)^2 - 4(1)(8x - 4) = 32 - 32x$ so

$$y = \frac{-4 \pm \sqrt{32 - 32x}}{2} = \frac{-4 \pm \sqrt{32(1 - x)}}{2} = \frac{-4 \pm 4\sqrt{2(1 - x)}}{2} = -2 \pm 2\sqrt{2 - 2x}.$$

We get two functions: $f(x) = -2 + 2\sqrt{2 - 2x}$ and $g(x) = -2 - 2\sqrt{2 - 2x}$. Since $\sqrt{2 - 2x} \geq 0$. The graph of f traces out the *upper* half of the parabola while the graph of g traces out the *lower* half of the parabola, as verified by a graphing utility.

3. Find the standard form of the parabola satisfying the following characteristics:

   (a) We begin by sketching the data given to us below.



We see immediately we have a vertical parabola here so we work to find an equation of the form:

$$(x - h)^2 = 4p(y - k).$$

Since the focus is $(2, 1)$, we know the the vertex lies on the vertical line $x = 2$. Moreover, since the vertex is halfway between the focus and directrix, we know the vertex is exactly $\frac{5}{2}$ units *below* the focus at $\left(2, -\frac{3}{2}\right)$. This gives $h = 2$ and $k = -\frac{3}{2}$. Since the focus of the parabola is $\frac{5}{2}$ units *above* the vertex we know $p = +\frac{5}{2}$. Our answer is:

$$(x - 2)^2 = 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right)^2 \quad \text{or} \quad (x - 2)^2 = 10\left(y + \frac{3}{2}\right)$$

(b) Since we have a horizontal parabola, we know our answer takes the form of $(y-k)^2 = 4p(x-h)$. From the graph, we may infer the vertex of the parabola is $(-2, 3)$ so $h = -2$ and $k = 3$. So we have $(y-3)^2 = 4p(x+2)$. Since the graph contains $(0, 6)$, we substitute $x = 0$ and $y = 6$ to get $(6-3)^2 = 4p(0+2)$. We find $p = \frac{9}{8}$ which gives our final answer

$$(y-3)^2 = 4\left(\frac{9}{8}\right)(x+2) \quad \text{or} \quad (y-3)^2 = \frac{9}{2}(x+2).$$